GEOMETRICAL DEMONSTRATION OF A THEOREM.

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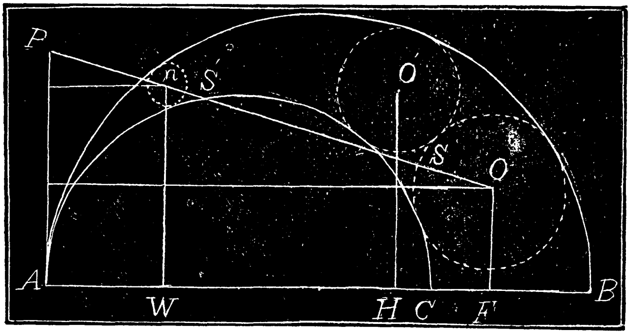

Let T be the common tangent, and d_1, d_3 the diameters of the two circles that can be described touching three circles that touch each other, then $T^2 = 4d_1d_3$.

This theorem is given by Matthew Collins, on page 278 of *Mathematical Monthly*, Vol. 1, who states that he proposed it in an old number of the "Educational Times," but no geometrical demonstation had yet appeared. He then gives a solution of a particular case, only, namely, that in which one of the three circles that thouch each other becomes infinite.

The following method pre-supposes some knowledge of the properties of Centers of Similitude, of radical axes and of circles in contact, and it is based on a beautiful theorem, remarkable for its generality, known to the ancients under the name of "The Arbelos," or "The Shoemaker's Knife," which is enunciated by Mr. Collins, in the paper referred to, by means of a figure, as follows:

"If the semicircles on the diameters AB, AC, touching each other at A, be both touched by the circles whose centers are O and O′. which touch each other at L; demit OF, O′H perpendiculars on AB, then if OF $=$ n times the diameter of O, O′H $=$ $(n + 1)$ times the diameter of O′, O′ being nearer io A than O is."

The elegant geometrical proof there given, holds true, if the two original semicircles touched each other externally at A.

Let the circle whose center is O″, touch the circle O′ and the two original semicircles, O‴ touch O″ and the semicircles, and so on in the same order, N being the center of the n^{th} circle from O; it is required to find the relation connecting the quantities T, R and r, T being the

common tangent, and R, r, the radii of the circles whose centers are O and N.

This may be easily accomplished by means of the Arbelos, thus:

Putting OF $=m$ times 2R, then NW $=(m+n).2r$.

$$\frac{NW}{2r}-\frac{OF}{2R}=n. \qquad NW.R-OF.r=2nRr \ldots\ldots\ldots(1).$$

Putting ON $=d$, $\qquad d^2=T^2+(R-r)^2 \ldots\ldots\ldots\ldots\ldots(2).$

Now produce ON until it intersects the common tangent, or radical axis of the original semicircles in P; hence by a well-kdown theorem, namely, "If each of two circles touch (in the same way) another pair of circles, the center of similitude of either pair lies upon the radical axis of the other pair," P is the external center of similitude of the circles O and N, and $\qquad \dfrac{PO}{PN}=\dfrac{R}{r} \ldots\ldots\ldots\ldots\ldots\ldots(3).$

Also S, S′, being two anti-homologous points, (see Chauvenet's, p. 360, or any treatise on the Modern Geometry) on the circles O and N, we have the equation

$$\sqrt{(PN+NS')(PO-OS)}=PA \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

From (3) we get $\qquad \dfrac{dr}{R-r}=PN,\ \dfrac{dR}{R-r}=PO.$

Hence (4) becomes

$$\sqrt{\left(\frac{dr}{R-r}+r\right)\left(\frac{dR}{R-r}-R\right)}=\sqrt{\frac{d^2Rr}{(R-r)^2}-Rr}=PA \ldots\ldots\ldots(5).$$

By similar triangles

$$\frac{PA-OF}{PA-NW}=\frac{PO}{PN}=\frac{R}{r}.$$

$PA(R-r)=NW.R-OF.r=2nRr$ from (1).

$PA=\dfrac{2nRr}{R-r}$. By substituting in eq. (5), this value of PA, and the value of d^2 as given in (2), we get

$$\left\{\frac{[T^2+(R-r)^2]Rr}{(R-r)^2}-Rr\right\}^{\frac{1}{2}}=\frac{2nRr}{R-r},$$

which reduces to $T = 2n\sqrt{Rr}$, the relation required; or putting $R = \frac{1}{2}d_1$, $r = \frac{1}{2}d_n$, we get $T = n\sqrt{d_1 d_n}$.

If $n = 2$, we have the original theorem proposed by Mr. Collins.

The same method will apply, if the two original semicircles touched each other externally at A.

If we consider T the transverse, instead of the direct common tangent of O and N, then (2) becomes

$$d^2 = T_1^2 + (R + r)^2.$$

Substitute in (5) as before and

$$T_1 = 2\left((n^2 - 1)Rr\right)^{\frac{1}{2}} = \sqrt{(n^2 - 1)d_1 d_n}.$$

In this case, if $n = 2$, $\qquad T_1^2 = 3d_1 d_n$.

SOLUTION OF A PROBLEM.

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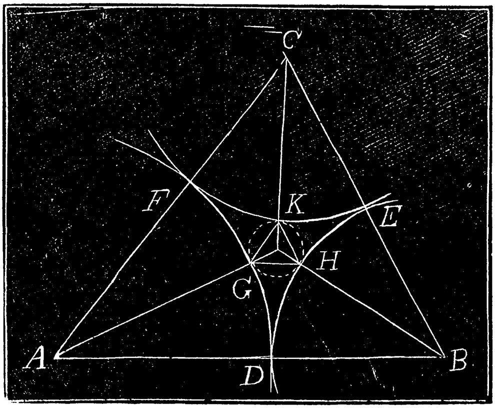
Three circles whose radii are a, b, c, touch each other externally. Within the space enclosed by them a circle is drawn tangent to the three circles, and within this circle three circles are drawn tangent to each other and to the three given circles. Calling the radii of these three circles x_1, y_1, z_1, we may determine three other circles, radii x_2, y_2, z_2, touching each other and the second set of circles in a similar way; and so on. Find the radii, x_n, y_n, z_n, of the n^{th} set of inscribed circles.



SOLUTION.—Let A, B, C be the centers of the given circles, GHK the circle which touches them, O its center, A′, B′, C′ the centers of the first set of inscribed circles.

Put $AD = a$, $BD = b$, $CF = c$, $OG = r_1$, $A'G = x_1$, $B'H = y_1$, $C'K = z_1$, r_0 = the radius of the circle circumscribing the three given circles; also put $\angle BAC = \beta$, $\angle BAO = \theta$, $\angle CAO = \psi$, $\angle AOB = \omega$.